

# QUASI-GRAPHIC MATROIDS

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**ABSTRACT.** Frame matroids and lifted-graphic matroids are two interesting generalizations of graphic matroids. Here we introduce a new generalization, *quasi-graphic matroids*, that unifies these two existing classes. Unlike frame matroids and lifted-graphic matroids, it is easy to certify that a matroid is quasi-graphic. The main result of the paper is that every 3-connected representable quasi-graphic matroid is either a lifted-graphic matroid or a frame matroid.

## 1. INTRODUCTION

Let  $G$  be a graph and let  $M$  be a matroid. For a vertex  $v$  of  $G$  we let  $\text{loops}_G(v)$  denote the set of loop-edges of  $G$  at the vertex  $v$ . We say that  $G$  is a *framework* for  $M$  if

- (1)  $E(G) = E(M)$ ,
- (2)  $r_M(E(H)) \leq |V(H)|$  for each component  $H$  of  $G$ , and
- (3) for each vertex  $v$  of  $G$  we have  $\text{cl}_M(E(G - v)) \subseteq E(G - v) \cup \text{loops}_G(v)$ .

This definition is motivated by the following result that is essentially due to Seymour [1].

**Theorem 1.1.** *Let  $G$  be a graph with  $c$  components and let  $M$  be a matroid. Then  $M$  is the cycle matroid of  $G$  if and only if  $G$  is a framework for  $M$  and  $r(M) \leq |V(G)| - c$ .*

We will call a matroid *quasi-graphic* if it has a framework. Next we will consider two classes of quasi-graphic matroids; namely “lifted-graphic matroids” and “frame matroids”.

We say that a matroid  $M$  is a *lift* of a matroid  $N$  if there is a matroid  $M'$  and an element  $e \in E(M')$  such that  $M' \setminus e = M$  and  $M'/e = N$ .

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*Date:* December 10, 2015.

*1991 Mathematics Subject Classification.* 05B35.

*Key words and phrases.* matroids, representation, graphic matroids, frame matroids.

This research was partially supported by grants from the Office of Naval Research [N00014-10-1-0851], NSERC [203110-2011], the Marsden Fund of New Zealand, and NWO (The Netherlands Organisation for Scientific Research).

If  $M$  is a lift of a graphic matroid, then we will call  $M$  a *lifted-graphic matroid*.

**Theorem 1.2.** *If  $G$  is a graph and  $M$  is a lift of  $M(G)$ , then  $G$  is a framework for  $M$ .*

We say that a matroid  $M$  is *framed* if it has a basis  $V$  such that for each element  $e \in E(M)$  there is a set  $W \subseteq V$  such that  $|W| \leq 2$  and  $e \in \text{cl}_M(W)$ . A *frame matroid* is a restriction of a framed matroid.

**Theorem 1.3.** *Every frame matroid is quasi-graphic.*

Our main result is that for matroids that are both 3-connected and representable, there are no kinds of quasi-graphic matroids other than those described above.

**Theorem 1.4.** *Let  $M$  be a 3-connected representable matroid. If  $M$  is quasi-graphic, then either  $M$  is a frame matroid or  $M$  is a lifted-graphic matroid.*

The representability condition in Theorem 1.4 is necessary; the Vámos matroid, for example, is quasi-graphic but it is neither a frame matroid nor a lifted-graphic matroid. However, for frameworks with loop-edges, we do not require representability.

**Theorem 1.5.** *Let  $G$  be a framework for a 3-connected matroid  $M$ . If  $G$  has a loop-edge, then  $M$  is either a frame matroid or a lifted-graphic matroid.*

Our proof of Theorem 1.5 uses results of Zaslavsky [2] who characterized frame matroids and lifted-graphic matroids using “biased graphs”; we review those results in Sections 4 and 5.

One attractive feature of frameworks is that they are easy to certify. That is, given a graph  $G$  and a matroid  $M$  one can readily check whether or not  $G$  is a framework for  $M$ . More specifically, there is a polynomial-time algorithm that given  $G$  and  $M$  (via its rank oracle) will decide whether or not  $G$  is a framework for  $M$ .

We conjecture that there is no general way for certifying that a matroid is a frame matroid, or a lifted-graphic matroid, using only polynomially many rank evaluations.

**Conjecture 1.6.** *For any polynomial  $p(\cdot)$  there is a frame matroid  $M$  such that for any set  $\mathcal{S}$  of subsets of  $E(M)$  with  $|\mathcal{S}| \leq p(|M|)$  there is a non-frame matroid  $M'$  such that  $E(M') = E(M)$  and  $r_{M'}(X) = r_M(X)$  for each  $X \in \mathcal{S}$ .*

**Conjecture 1.7.** *For any polynomial  $p(\cdot)$  there is a lifted-graphic matroid  $M$  such that for any set  $\mathcal{S}$  of subsets of  $E(M)$  with  $|\mathcal{S}| \leq p(|M|)$  there is a non-lifted-graphic matroid  $M'$  such that  $E(M') = E(M)$  and  $r_{M'}(X) = r_M(X)$  for each  $X \in \mathcal{S}$ .*

In stark contrast to these two negative conjectures, we conjecture that the problem of recognizing quasi-graphic matroids is tractable.

**Conjecture 1.8.** *There is a polynomial-time algorithm that given a matroid  $M$ , via its rank-oracle, decides whether or not  $M$  is quasi-graphic.*

## 2. MINORS OF QUASI-GRAPHIC MATROIDS

We will start by proving that the class of quasi-graphic matroids is minor-closed.

**Lemma 2.1.** *Let  $G$  be a framework for  $M$ . If  $H$  is a component of  $G$ , then  $H$  is a framework for  $M|E(H)$ .*

*Proof.* Note that conditions (1) and (2) are immediate. Condition (3) follows from the fact that for each flat  $F$  of  $M$ , the set  $F \cap E(H)$  is a flat of  $M|E(H)$ .  $\square$

The following result is very easy, but it is used repeatedly.

**Lemma 2.2.** *Let  $G$  be a framework for  $M$ . If  $v$  is a vertex of  $G$  that is incident with at least one non-loop-edge, then  $r_M(E(G - v)) < r(M)$ . Moreover, if  $v$  has degree one, then  $r_M(E(G - v)) = r(M) - 1$ .*

*Proof.* This follows directly from (3).  $\square$

**Lemma 2.3.** *Let  $G$  be a connected framework for  $M$  and let  $H$  be a subgraph of  $G$ . Then  $|V(H)| - r(M|E(H)) \geq |V(G)| - r(M)$ .*

*Proof.* The result holds when  $H$  is trivial, so we may assume that  $V(H) \neq \emptyset$ . We can extend  $H$  to a spanning subgraph  $H^+$  of  $G$  with  $|E(H^+)| - |E(H)| = |V(G)| - |V(H)|$ . Clearly  $|V(H^+)| - r(E(H^+)) \geq |V(G)| - r(M)$ . If  $H \neq H^+$ , then there is a vertex  $v \in V(H^+) - V(H)$  that has degree one in  $H^+$ . By Lemma 2.2,  $r(E(H^+ - v)) = r(E(H)) - 1$  and, hence,  $|V(H^+ - v)| - r(E(H^+ - v)) \geq |V(G)| - r(M)$ . Now we obtain the result by repeatedly deleting vertices in  $V(H^+) - V(H)$  in this way.  $\square$

If  $X$  is a set of edges in a graph  $G$ , then  $G[X]$  is the subgraph of  $G$  with edge-set  $X$  and with no isolated vertices.

**Lemma 2.4.** *Let  $G$  be a framework for  $M$  and let  $X \subseteq E(M)$ . Then  $G[X]$  is a framework for  $M|X$ .*

*Proof.* Condition (1) is clearly satisfied. Condition (2) follows from Lemmas 2.1 and 2.3. Condition (3) follows from the fact that for each flat  $F$  of  $M$ , the set  $F \cap E(H)$  is a flat of  $M|E(H)$ .  $\square$

The following two results give sufficient conditions for independence and dependence, respectively, for a set in a matroid given only the structure in the framework.

**Lemma 2.5.** *Let  $G$  be a framework for  $M$ . If  $F$  is a forest of  $G$ , then  $E(F)$  is an independent set of  $M$ .*

*Proof.* We may assume that  $E(F)$  is non-empty and, hence, that  $F$  has a degree-one vertex  $v$ . By Lemma 2.2,  $r_M(E(F)) = r_M(E(F - v)) + 1$ . Now the result follows inductively.  $\square$

**Lemma 2.6.** *Let  $G$  be a framework for  $M$ . If  $H$  is a subgraph of  $G$  and  $|E(H)| > |V(H)|$ , then  $E(H)$  is a dependent set of  $M$ .*

*Proof.* By Lemma 2.4 and (2), we have  $r_M(E(H)) \leq |V(H)|$ . So, if  $|E(H)| > |V(H)|$ , then  $E(H)$  is a dependent set of  $M$ .  $\square$

We can now prove Theorem 1.1.

**Theorem** (Theorem 1.1 restated). *Let  $G$  be a graph with  $c$  components and let  $M$  be a matroid. Then  $M$  is the cycle matroid of  $G$  if and only if  $G$  is a framework for  $M$  and  $r(M) \leq |V(G)| - c$ .*

*Proof.* By Lemma 2.5 and the fact that  $r(M) \leq |V(G)| - c$ , we have  $r(E(H)) = |V(H)| - 1$  for each component  $H$  of  $G$ . Hence we may assume that  $G$  is connected. By Lemma 2.5, the edge-set of each forest of  $G$  is independent in  $M$ . Therefore, it suffices to prove, for each circuit  $C$  of  $G$ , that  $E(C)$  is dependent in  $M$ . By Lemma 2.3,  $|V(C)| - r(E(C)) \geq |V(G)| - r(E(G)) = 1$ . So  $r(E(C)) < |V(C)| = |E(C)|$  and, hence,  $E(C)$  is dependent as required.  $\square$

To prove that the class of quasi-graphic matroids is closed under contraction, we consider two cases depending on whether or not we are contracting a loop-edge of the framework.

**Lemma 2.7.** *Let  $G$  be a framework for  $M$  and let  $e$  be a non-loop-edge of  $G$ . Then  $G/e$  is a framework for  $M/e$ .*

*Proof.* Conditions (1) and (2) are clearly satisfied. Let  $u$  and  $v$  be the ends of  $e$  in  $G$ , and let  $f$  be an edge of  $G$  that is incident with  $u$  but not with  $v$ . To prove (3) it suffices to prove that there exists a cocircuit  $C$  in  $M$  such that  $f \in C$ ,  $e \notin C$ , and  $C$  contains only edges incident with either  $u$  or  $v$ .

By (3), there exist cocircuits  $C_e$  and  $C_f$  such that  $e \in C_e$ , that  $C_e$  contains only edges incident with  $v$ , that  $f \in C_f$ , and that  $C_f$  contains only edges incident with  $u$ . We may assume that  $e \in C_f$  since otherwise we could take  $C = C_f$ . Since  $f$  is not incident with  $v$ , we have  $f \notin C_e$ . Then, by the strong circuit exchange axiom, there is a cocircuit  $C$  of  $M$  with  $f \in C \subseteq (C_1 \cup C_2) - \{e\}$ , as required.  $\square$

**Lemma 2.8.** *Let  $G$  be a framework for  $M$ , let  $e$  be a loop-edge of  $G$  at a vertex  $v$  and let  $H$  be the graph obtained by first, for each non-loop edge  $f = vw$  incident with  $v$  adding  $f$  as a loop at  $w$ , and then for each loop-edge  $f$  of  $G - e$  at  $v$  adding  $f$  as a loop on an arbitrary vertex. If  $e$  is not a loop of  $M$ , then  $H$  is a framework for  $M/e$ .*

*Proof.* Conditions (1) and (2) are clearly satisfied. By Lemma 2.4, we have  $r_M(\text{loops}_G(v)) = 1$ , so each element of  $\text{loops}_G(v) - \{e\}$  is a loop in  $M/e$ . Each vertex  $w \in V(G) - \{v\}$  is incident with the same edges in  $G$  as it is in  $H$  except for the elements in  $\text{loops}_G(v)$ . Moreover,  $\text{cl}_M(E(G - w)) = \text{cl}_{M/e}(E(H - w)) \cup \{e\}$ . Therefore (3) follows.  $\square$

We have proved the following:

**Theorem 2.9.** *The class of quasi-graphic matroids is closed under taking minors.*

### 3. BALANCED CIRCUITS

Let  $G$  be a framework for a matroid  $M$ . If  $C$  is a circuit of  $G$ , then, by Lemmas 2.3 and 2.5,  $E(C)$  is either independent in  $M$  or  $E(C)$  is a circuit in  $M$ ; we say that  $C$  is *balanced* if  $E(C)$  is a circuit of  $M$ .

**Lemma 3.1.** *Let  $G$  be a framework for  $M$ . Then  $M = M(G)$  if and only if each circuit of  $G$  is balanced.*

*Proof.* If  $M = M(G)$ , then each circuit of  $G$  is balanced. Conversely, suppose that each circuit of  $G$  is balanced. Let  $F$  be a maximal forest in  $G$ . Since each circuit is balanced,  $E(F)$  is a basis of  $M$ . Then, by Theorem 1.1,  $M = M(G)$ .  $\square$

A *theta* is a 2-connected graph that has exactly two vertices of degree 3 and all other vertices have degree 2. Observe that there are exactly three circuits in a theta.

**Lemma 3.2.** *Let  $G$  be a framework for  $M$  and let  $H$  be a theta-subgraph of  $G$ . If two of the circuits in  $H$  are balanced, then so too is the third.*

*Proof.* If there are two balanced circuits in  $H$  then  $r_M(E(H)) \leq |E(H)| - 2 = |V(H)| - 1$ . So, by Theorem 1.1,  $M|E(H) = M(H)$  and, by Lemma 3.1, all circuits of  $H$  are balanced.  $\square$

The following result describes the circuits of a matroid in terms of the framework; first we will give an unusual example to demonstrate one of the outcomes. If  $M$  consists of a single circuit and  $G$  is a graph with  $E(G) = E(M)$  whose components are circuits, then  $G$  is a framework for  $M$ .

**Lemma 3.3.** *Let  $G$  be a framework for  $M$  and let  $C$  be a circuit in  $M$ . Then either*

- $G[C]$  is a balanced circuit,
- $G[C]$  is a connected graph with minimum degree at least two,  $|C| = V(G[C]) + 1$ , and  $G[C]$  has no balanced circuits, or
- $G[C]$  is a collection of vertex-disjoint non-balanced circuits.

*Proof.* We may assume that  $G[C]$  is not a balanced circuit, and, hence, that  $G[C]$  contains no balanced circuit. Next suppose that  $|C| \geq V(G[C]) + 1$ . By Lemma 2.6,  $C$  is minimal with this property. Hence  $G[C]$  is connected, the minimum degree of  $G[C]$  is two, and  $|C| = V(G[C]) + 1$ . Now suppose that  $|C| \leq V(G[C])$  and consider a component  $H$  of  $G[C]$ ; it suffices to show that  $G[C]$  is a circuit. By Lemma 2.6 and the argument above, we may assume that  $|E(H)| \leq |V(H)|$ . If  $H$  is not a circuit there is a degree-one vertex  $v$  of  $H$ . Moreover, the edge  $e$  that is incident with  $v$  is not a loop-edge. Then, by (3), the element  $e$  is a coloop of  $M|C$ , which contradicts the fact that  $C$  is a circuit.  $\square$

For a set  $X$  of elements in a matroid  $M$  we let

$$\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M).$$

**Lemma 3.4.** *Let  $G$  be a framework for  $M$ . If  $H$  is a component of  $G$ , then  $\lambda_M(E(H)) \leq 1$ .*

*Proof.* By Lemma 2.2,  $r(E(M) - E(H)) \leq r(M) - (|V(H)| - 1)$ . Hence  $\lambda_M(E(H)) = r_M(E(H)) + r_M(E(M) - E(H)) - r(M) \leq |V(H)| + (r(M) - (|V(H)| - 1)) - r(M) = 1$ .  $\square$

The following result is an immediate consequence of Lemma 3.4.

**Lemma 3.5.** *If  $G$  is a framework for a 3-connected matroid  $M$  with  $|M| \geq 4$  and  $G$  has no isolated vertices, then either*

- $G$  is connected, or
- $G$  has two components one of which consists of a single vertex with a loop.

**Lemma 3.6.** *Let  $M$  be a 3-connected matroid with  $|E(M)| \geq 4$ . If  $M$  is quasi-graphic, then  $M$  has a connected framework.*

*Proof.* Let  $G$  be a framework for  $M$  and suppose that  $G$  is not connected. We may assume that  $G$  has no isolated vertices. Then, by Lemma 3.5,  $G$  has two components, one of which consists of a single vertex  $v$  and a single edge  $e$ . Since  $e$  is not a coloop of  $M$ ,  $r(M) \leq |V(G)| - 1$ . Let  $w \in V(G) - \{v\}$ . Now we construct a new graph  $G^+$  by adding a new edge  $f$  with ends  $v$  and  $w$  and let  $M^+$  be a matroid obtained from  $M$  by adding  $f$  as a coloop. Note that  $G^+$  is a framework for  $M^+$ . Therefore  $G^+/f$  is a framework for  $M^+/f$ . Since  $f$  is a coloop of  $M^+$ , we have  $M^+/f = M^+ \setminus f = M$ . So  $G^+/f$  is a connected framework for  $M$ .  $\square$

**Lemma 3.7.** *Let  $M$  be a 3-connected matroid with  $|E(M)| \geq 4$ . If  $G$  is a connected framework for  $M$ , then  $G$  is 2-connected.*

*Proof.* Suppose otherwise. Then there is a pair  $(H_1, H_2)$  of subgraphs of  $G$  such that  $G = H_1 \cup H_2$ ,  $|V(H_1) \cap V(H_2)| = 1$ , and  $|V(H_1)|, |V(H_2)| \geq 2$ . Note that  $H_1$  and  $H_2$  are both connected. Now  $M(G)$  is not 3-connected, so, by Lemma 1.1,  $r(M) = |V(G)|$ . Therefore  $\lambda_M(E(H_1)) \leq |V(H_1)| + |V(H_2)| - |V(G)| = 1$ . Since  $M$  is 3-connected either  $|E(H_1)| \leq 1$  or  $|E(H_2)| \leq 1$ ; we may assume that  $|E(H_1)| = 1$ . Let  $e \in E(H_1)$ . Since  $H_1$  is a connected and  $|V(H_1)| \geq 2$ , the edge  $e$  is not a loop. Therefore, by (3),  $e$  is a coloop of  $M$ . This contradicts the fact that  $M$  is 3-connected.  $\square$

The following two lemmas refine Lemma 3.3 in the case that  $M$  is 3-connected.

**Lemma 3.8.** *Let  $M$  be a 3-connected matroid with  $|M| \geq 4$  and let  $G$  be a framework for  $M$ . If  $C_1$  and  $C_2$  are vertex-disjoint non-balanced circuits of  $G$ , then either*

- $E(C_1) \cup E(C_2)$  is a circuit of  $M$ ,
- $E(C_1) \cup E(C_2) \cup E(P)$  is a circuit of  $M$  for each minimal path  $P$  in  $G$  from  $V(C_1)$  to  $V(C_2)$ .

*Moreover, if  $C_1$  and  $C_2$  are in distinct components of  $G$ , then  $E(C_1) \cup E(C_2)$  is a circuit of  $M$ .*

*Proof.* We may assume that  $E(C_1) \cup E(C_2)$  is not a circuit. Let  $P$  be a minimal path in  $G$  from  $V(C_1)$  to  $V(C_2)$ . By Lemma 2.6,  $E(C_1 \cup C_2 \cup P)$  is dependent. Let  $C \subseteq E(C_1 \cup C_2 \cup P)$  be a circuit of  $M$ . By Lemma 3.3,  $C = E(C_1 \cup C_2 \cup P)$ .

Finally, suppose that  $C_1$  and  $C_2$  are in distinct components of  $G$ . We may assume that  $G$  has no isolated vertices. Then, by Lemma 3.5,  $G$  has two components one of which has a single vertex, say  $v$ , and a single loop-edge, say  $e$ . Since  $M$  is 3-connected,  $e$  is not a coloop of

$M$ . Then, by (3),  $r(M) \leq |V(G)|$ . We may assume that  $E(C_1) = \{e\}$ ; let  $w$  be a vertex of  $C_2$ . Construct a graph  $G^+$  from  $G$  by adding a new edge  $f$  with ends  $v$  and  $w$  and construct a new matroid  $M^+$  by adding  $f$  as a coloop to  $M$ . Note that  $G^+$  is a framework for  $M^+$  and hence  $G^+/f$  is a framework for  $M^+/f$ . By Lemmas 2.6 and 3.3,  $E(C_1) \cup E(C_2)$  is a circuit in  $M^+/f$ . Moreover, as  $f$  is a coloop of  $M^+$ , we have  $M^+/f = M$ , so  $E(C_1) \cup E(C_2)$  is a circuit in  $M$ .  $\square$

**Lemma 3.9.** *Let  $M$  be a 3-connected matroid with  $|M| \geq 4$  and let  $G$  be a framework for  $M$ . If  $C$  is a circuit for  $M$ , then  $G[C]$  has at most two components.*

*Proof.* Suppose that  $G[C]$  has more than two components. By Lemma 3.3, each component of  $G$  is a balanced circuit. By Lemma 3.5, two of these circuits are in the same component of  $G$ . Let  $P$  be a shortest path connecting two components of  $G[C]$ ; let these components be  $C_1$  and  $C_2$ . Since  $C$  is a circuit,  $G[C_1 \cup C_2]$  is independent. Therefore, by Lemma 3.8,  $E(C_1 \cup C_2 \cup P)$  is a circuit of  $M$ . Let  $e \in E(P)$  and  $f \in E(C_1)$ . By the strong exchange property for circuits, there is a circuit  $C'$  of  $G$  with  $e \in C' \subseteq (C \cup E(P)) - \{f\}$ . However this is inconsistent with the outcomes of Lemma 3.3.  $\square$

#### 4. FRAME MATROIDS

We start by proving Theorem 1.3.

**Theorem** (Theorem 1.3 restated). *Every frame matroid is quasi-graphic.*

*Proof.* Let  $M$  be a frame matroid. Note that  $M$  is a quasi-graphic matroid if and only if  $\text{si}(M)$  is a quasi-graphic matroid, so we may assume that  $M$  is simple. Recall that the class of quasi-graphic matroids is closed under taking minors, so we may further assume that  $M$  is framed; let  $V$  be a basis of  $M$  such that each element is spanned by a 2-element subset of  $V$ . We now construct a graph  $G$  with vertex-set  $V$  and edge-set  $E(M)$  such that, for each  $v \in V$  the edge  $v$  is a loop on the vertex  $v$  and for each  $e \in E(M) - V$  the edge  $e$  has ends  $u$  and  $v$  where  $\{e, u, v\}$  is the unique circuit of  $M$  in  $V \cup \{e\}$ . We claim that  $G$  is a framework for  $M$ .

By construction  $E(G) = E(M)$  and, since  $V$  is a basis of  $M$ , for each component  $H$  of  $G$  we have  $r(E(H)) = |V(H)|$ . Finally, for each vertex  $v$  of  $G$ , the hyperplane of  $M$  spanned by  $V - \{v\}$  is  $E(G - v)$ . Hence  $G$  is indeed a framework for  $M$ .  $\square$

Next we give an alternative characterization of frame matroids using frameworks; these results are effectively due to Zaslavsky [2].



Let  $G$  be a graph and let  $\mathcal{B}$  be a subset of the circuits of  $G$ . We say that  $\mathcal{B}$  satisfies the *theta-property* if there is no theta in  $G$  with exactly two of its three circuits in  $\mathcal{B}$ .

**Theorem 4.1.** *Let  $G$  be a graph and let  $\mathcal{B}$  be a collection of circuits in  $G$  that satisfy the theta-property. Now let  $M = (E(G), \mathcal{I})$  where a set  $I \subseteq E(G)$  is contained in  $\mathcal{I}$  if and only if there is no  $C \in \mathcal{B}$  with  $E(C) \subseteq I$  and  $|E(H)| \leq |V(H)|$  for each component  $H$  of  $G[I]$ . Then  $M$  is a matroid.*

*Proof.* We call the circuits of  $G$  in  $\mathcal{B}$  *balanced*. To prove that  $M$  is a matroid it suffices to check the following conditions, which are effectively a reformulation of the circuit axioms in terms of independent sets:

- (a)  $\emptyset \in \mathcal{I}$ ,
- (b) for each  $J \in \mathcal{I}$  and  $I \subseteq J$ , we have  $I \in \mathcal{I}$ , and
- (c) for each set  $I \in \mathcal{I}$  and  $e \in E(M) - I$  either  $I \cup \{e\} \in \mathcal{I}$  or there is a unique minimal subset  $C$  of  $I \cup \{e\}$  that is not in  $\mathcal{I}$ .

Conditions (a) and (b) follow from the construction.

Let  $I \in \mathcal{I}$  and  $e \in E(M) - I$  with  $I \cup \{e\} \notin \mathcal{I}$ . Let  $C_1$  and  $C_2$  be minimal subsets of  $I \cup \{e\}$  that are not in  $\mathcal{I}$ . Suppose for a contradiction that  $C_1 \neq C_2$ . By definition, for each  $i \in \{1, 2\}$ , we have  $G[C_i - \{e\}]$  is connected,  $e \in C_i$ , and either  $G[C_i]$  is a balanced circuit or  $|C_i| > |V(G[C_i])|$ . Consider  $J = (C_1 \cup C_2) - \{e\}$ . Since  $J \subseteq I$ , we have  $J \in \mathcal{I}$ . Since  $G[C_1 - \{e\}]$  and  $G[C_2 - \{e\}]$  are connected,  $G[J]$  is connected. Therefore  $|J| \leq |V(G[J])|$ . It follows that  $|C_1| \leq |V(G[C_1])|$  and  $|C_2| \leq |V(G[C_2])|$ . Hence  $G[C_1]$  and  $G[C_2]$  are balanced circuits. Now  $G[C_1 \cup C_2]$  is a theta and  $G[J]$  is a circuit. By the theta-property,  $G[J]$  is balanced. However, this contradicts the fact that  $J \in \mathcal{I}$ .  $\square$

We denote the matroid  $M$  in Theorem 4.1 by  $FM(G, \mathcal{B})$ .

**Theorem 4.2.** *If  $G$  is a graph and  $\mathcal{B}$  is a collection of circuits in  $G$  that satisfies the theta-property, then  $FM(G, \mathcal{B})$  is a frame matroid.*

*Proof.* Let  $G^+$  be obtained from  $G$  by adding a loop-edge  $e_v$  at each vertex of  $v$ . Now let  $\mathcal{B}^+$  be obtained from  $\mathcal{B}$  by adding the circuits  $(G[\{e_v\}] : v \in V(G))$ . Since we only added loops to  $\mathcal{B}$ , the collection  $\mathcal{B}^+$  satisfies the theta-property. Let  $M^+ = FM(G^+, \mathcal{B}^+)$  and  $V = \{e_v : v \in V(G)\}$ . By the definition of  $FM(G^+, \mathcal{B}^+)$ , the set  $V$  is a basis of  $M^+$ . For each non-loop edge  $e$  of  $G$  with ends  $u$  and  $v$ , the set  $\{e_u, e, e_v\}$  is a circuit of  $M^+$  and for each loop-edge  $e$  of  $G$  at  $v$ , the set  $\{e, e_v\}$  is a circuit of  $M^+$ . Therefore  $M^+$  is a framed matroid and hence  $FM(G, \mathcal{B})$  is a frame matroid.  $\square$

**Theorem 4.3.** *A loopless matroid  $M$  is a frame matroid if and only if there is a graph  $G$  and a collection  $\mathcal{B}$  of circuits of  $G$  satisfying the theta-property such that  $M = FM(G, \mathcal{B})$ .*

*Proof.* The “if” direction of the result follows from Theorem 4.2. For the converse we may assume that  $M$  is a framed matroid; let  $V$  be a basis of  $M$  such that each element is spanned by a 2-element subset of  $V$ . We now construct a graph  $G$  with vertex-set  $V$  and edge-set  $E(M)$  such that, for each  $v \in V$  the edge  $v$  is a loop on the vertex  $v$  and for each  $e \in E(M) - V$  the edge  $e$  has ends  $u$  and  $v$  where  $\{e, u, v\}$  is the unique circuit of  $M$  in  $V \cup \{e\}$ . By the proof of Theorem 1.3,  $G$  is a framework for  $M$ .

By Lemma 3.3, it suffices to prove that, if  $C_1, \dots, C_k$  are disjoint non-balanced circuits of  $G$ , then  $E(C_1 \cup \dots \cup C_k)$  is independent. This follows from the fact that  $V(C_1 \cup \dots \cup C_k)$  is independent and that, for each  $i \in \{1, \dots, k\}$ , the sets  $E(C_i)$  and  $V(C_i)$  span each other.  $\square$

## 5. LIFTED-GRAPHIC MATROIDS

We start by proving Theorem 1.2.

**Theorem** (Theorem 1.2 restated). *If  $G$  is a graph and  $M$  is a lift of  $M(G)$ , then  $G$  is a framework for  $M$ .*

*Proof.* Let  $e$  be an element of a matroid  $M'$  such that  $M' \setminus e = M$  and  $M'/e = M(G)$ . Thus  $E(M) = E(G)$ . For each component  $H$  of  $G$ ,  $r_{M'/e}(E(H)) = |V(H)| - 1$  so  $r_M(E(H)) = r_{M'}(E(H)) \leq r_{M'/e}(E(H)) + 1 = |V(H)|$ . For a vertex  $v$  of  $G$ , we have  $\text{cl}_M(E(G - v)) \subseteq \text{cl}_{M'}(E(G - v)) \subseteq \text{cl}_{M'/e}(E(G - v)) \cup \{e\} = \text{cl}_{M'/e}(E(G - v)) \subseteq E(G - v) \cup \text{loops}_G(v)$ . So  $G$  is a framework for  $M$ .  $\square$

Next we will give an alternative characterization of lifted-graphic matroids using frameworks; again, these results are effectively due to Zaslavsky [2].

**Theorem 5.1.** *Let  $G$  be a graph and let  $\mathcal{B}$  be a collection of circuits in  $G$  that satisfy the theta-property. Now let  $M = (E(G), \mathcal{I})$  where a set  $I \subseteq E(G)$  is contained in  $\mathcal{I}$  if and only if there is no  $C \in \mathcal{B}$  with  $E(C) \subseteq I$  and  $G[I]$  contains at most one circuit. Then  $M$  is a matroid and  $G$  is a framework for  $M$ .*

*Proof.* We call the circuits of  $G$  in  $\mathcal{B}$  *balanced*. To prove that  $M$  is a matroid it suffices to check the following conditions:

- (a)  $\emptyset \in \mathcal{I}$ ,
- (b) for each  $J \in \mathcal{I}$  and  $I \subseteq J$ , we have  $I \in \mathcal{I}$ , and

- (c) for each set  $I \in \mathcal{I}$  and  $e \in E(M) - I$  either  $I \cup \{e\} \in \mathcal{I}$  or there is a unique minimal subset  $C$  of  $I \cup \{e\}$  that is not in  $\mathcal{I}$ .

Conditions (a) and (b) follow from the construction.

Let  $I \in \mathcal{I}$  and  $e \in E(M) - I$  with  $I \cup \{e\} \notin \mathcal{I}$ . Let  $C_1$  and  $C_2$  be minimal subsets of  $I \cup \{e\}$  that are not in  $\mathcal{I}$ . Suppose for a contradiction that  $C_1 \neq C_2$ . By definition, for each  $i \in \{1, 2\}$ , either  $G[C_i]$  is a balanced circuit,  $G[C_i]$  is the union of two vertex disjoint non-balanced circuits, or  $G[C_i]$  is 2-edge-connected and  $|C_i| = |V(G[C_i])| + 1$ . Consider  $J = (C_1 \cup C_2) - \{e\}$ . Since  $J \subseteq I$ , we have  $J \in \mathcal{I}$  so either  $G[J]$  is a forest or  $G[J]$  contains a unique circuit.

For each  $i \in \{1, 2\}$ , there is a circuit  $A_i$  of  $G[C_i]$  that contains  $e$ . Since  $G[J]$  contains at most one circuit, either  $A_1 = A_2$  or  $A_1 \cup A_2$  is a theta.

First suppose that  $A_1 = A_2$ . Since  $C_1 \neq C_2$ , the circuit  $A_1$  is non-balanced. Therefore, for each  $i \in \{1, 2\}$ , there is a non-balanced circuit  $B_i$  in  $G[C_i - e]$ . Since  $G[J]$  contains a unique circuit  $B_1 = B_2$ . But then  $C_1 = E(A_1 \cup B_1)$  and  $C_2 = E(A_2 \cup B_2)$ , contradicting the fact that  $C_1 \neq C_2$ .

Now suppose that  $A_1 \cup A_2$  is a theta, and let  $C$  be the circuit in  $(A_1 \cup A_2) - e$ . Since  $J$  is independent,  $C$  is not balanced. By the theta-property and symmetry, we may assume that  $A_1$  is not balanced. Then there is a non-balanced circuit  $B_1$  in  $G[C_1 - \{e\}]$ . Since  $G[J]$  has at most one circuit  $C = B_1$ . Therefore  $C_1 = E(A_1 \cup A_2)$  and, hence,  $A_2$  is non-balanced. Then there is a non-balanced circuit  $B_2$  in  $G[C_2 - \{e\}]$ . Since  $G[J]$  has at most one circuit  $C = B_2$ , however, this contradicts the fact that  $C_1 \neq C_2$ .  $\square$

We denote the matroid  $M$  in Theorem 5.1 by  $LM(G, \mathcal{B})$ .

**Theorem 5.2.** *If  $G$  is a graph and  $\mathcal{B}$  is a collection of circuits in  $G$  that satisfies the theta-property, then  $LM(G, \mathcal{B})$  is a lift of  $M(G)$ .*

*Proof.* Let  $G^+$  be obtained from  $G$  by adding a loop-edge  $e$  at a vertex  $v$  and let  $\mathcal{B}^+ = \mathcal{B} \cup \{G[\{e\}]\}$ . Since we only added a loop to  $\mathcal{B}$ , the collection  $\mathcal{B}^+$  satisfies the theta-property. Let  $M^+ = LM(G^+, \mathcal{B}^+)$ . By the definition of  $LM(G^+, \mathcal{B}^+)$ , for each circuit  $C$  of  $G$ ,  $\{e\} \cup E(C)$  is dependent in  $M^+$ . Hence  $E(C)$  is a dependent set  $M^+/e$ . Similarly, by the definition of  $LM(G^+, \mathcal{B}^+)$ , for each forest  $F$  of  $G$ , the set  $\{e\} \cup E(F)$  is independent in  $M^+$  and, hence,  $E(F)$  is independent in  $M^+/e$ . Thus  $M^+/e = M(G)$  and, hence,  $M$  is a lift of  $M(G)$ .  $\square$

The following result is a converse to Theorem 5.2.

**Theorem 5.3.** *If  $G$  is a graph,  $M$  is a lift of  $M(G)$ , and  $\mathcal{B}$  is the set of balanced circuits of  $(M, G)$ , then  $M = LM(G, \mathcal{B})$ .*

*Proof.* It suffices to prove that if  $C_1$  and  $C_2$  are vertex disjoint circuits of  $G$ , then  $E(C_1 \cup C_2)$  is dependent in  $M$ . Now  $E(C_1 \cup C_2)$  has rank equal to  $|E(C_1 \cup C_2)| - 2$  in  $M(G)$  so its rank in  $M$  is at most  $|E(C_1 \cup C_2)| - 1$ . Thus  $E(C_1 \cup C_2)$  is indeed dependent in  $M$ .  $\square$

## 6. FRAMEWORKS WITH LOOPS

In this section we prove Theorem 1.5 which is an immediate consequence of the following two results.

**Theorem 6.1.** *Let  $G$  be a framework for a 3-connected matroid  $M$ , let  $\mathcal{B}$  be the set of balanced circuits of  $G$ , and let  $e$  be a non-balanced loop-edge at a vertex  $v$ . If  $e \in \text{cl}_M(E(G - v))$ , then  $M = LM(G, \mathcal{B})$ .*

*Proof.* It suffices to prove that if  $C_1$  and  $C_2$  are vertex-disjoint circuits of  $G$ , then  $E(C_1 \cup C_2)$  is dependent in  $M$ . We may assume that  $C_1$  and  $C_2$  are non-balanced and, by Lemma 3.8, we may assume that  $C_1$  and  $C_2$  are in the same component of  $G$ .

First suppose that  $C_1 = \{e\}$ . Let  $P$  be a minimal path from  $V(C_1)$  to  $V(C_2)$ . Let  $f$  be the edge of  $P$  that is incident with  $v$ . By (3) and the fact that  $e \in \text{cl}_M(E(G - v))$ , there is a cocircuit  $C^*$  of  $M$  such that  $C^* \cap E(C_1 \cup P \cup C_2) = \{f\}$ . Therefore  $E(C_1 \cup P \cup C_2)$  is not a circuit of  $M$ . So, by Lemma 3.8,  $E(C_1 \cup C_2)$  is a circuit of  $M$ , as required.

Now we may assume that neither  $C_1$  nor  $C_2$  is equal to  $G[\{e\}]$ . By the preceding paragraph, both  $E(C_1) \cup \{e\}$  and  $E(C_2) \cup \{e\}$  are circuits of  $M$ . So, by the circuit-exchange property,  $E(C_1 \cup C_2)$  is dependent, as required.  $\square$

**Theorem 6.2.** *Let  $G$  be a framework for a 3-connected matroid  $M$ , let  $\mathcal{B}$  be the set of balanced circuits of  $G$ , and let  $e$  be a loop-edge at a vertex  $v$ . If  $e \notin \text{cl}_M(E(G - v))$ , then  $M = FM(G, \mathcal{B})$ .*

*Proof.* By Lemmas 3.3, 3.8, and 3.9, it suffices to prove that, if  $C_1$  and  $C_2$  are vertex-disjoint non-balanced circuits of  $M$ , then  $E(C_1 \cup C_2)$  is independent in  $M$ .

First suppose that  $C_1 = G[\{e\}]$ . Since  $e \notin \text{cl}_M(E(G - v))$ , there is a cocircuit  $C^*$  of  $M$  that is disjoint from  $E(C_2)$ . Hence  $E(C_1 \cup C_2)$  is independent as required.

Now suppose the neither  $C_1$  nor  $C_2$  is equal to  $G[\{e\}]$ . We may also assume that  $E(C_1 \cup C_2)$  is dependent; by Lemma 3.3,  $E(C_1 \cup C_2)$  is a circuit of  $M$ . Since  $e \notin \text{cl}_M(E(G - v))$  and  $M$  has no co-loops,  $G[\{e\}]$  is not a component of  $G$ . Then, by Lemma 3.5, there is a path from  $v$  to  $V(C_1 \cup C_2)$  in  $G$ ; let  $P$  be a minimal such path. We may assume that  $P$  has an end in  $V(C_1)$ . By Lemma 3.8 and the preceding paragraph,  $E(C_1 \cup P) \cup \{e\}$  is a circuit of  $M$ . Let  $f \in E(C_1)$ ; by the circuit exchange

property, there exists a circuit  $C$  in  $(E(C_1 \cup C_2 \cup P) \cup \{e\}) - \{f\}$ . By Lemma 3.3,  $C = E(C_2) \cup \{e\}$ . However this contradicts the fact that  $e \notin \text{cl}_M(E(G - v))$ .  $\square$

## 7. REPRESENTABLE MATROIDS

A framework  $G$  for a matroid  $M$  is called *strong* if  $G$  is connected and  $r_M(E(G - v)) = r(M) - 1$  for each vertex  $v$  of  $G$ .

**Lemma 7.1.** *If  $M$  is a quasi-graphic matroid with  $|M| \geq 4$ , then  $M$  has a strong framework.*

*Proof.* By Lemma 3.6,  $M$  has a connected framework. Let  $G$  be a connected framework having as many loop-edges as possible. Suppose that  $G$  is not a strong framework and let  $v \in V(G)$  such that  $r_M(E(G) - v) < r(M) - 1$ . Let  $C^*$  be a cocircuit of  $M$  with  $C^* \cap E(G - v) = \emptyset$ ; if possible we choose  $C^*$  so that it contains a loop-edge of  $G$ . Since  $M$  is 3-connected,  $|C^*| \geq 2$  and, by Lemma 2.6, there is at most one loop-edge at  $v$ . Therefore  $C^*$  contains at least one non-loop-edge. Let  $L$  denote the set of non-loop-edges of  $G - C^*$  incident with  $v$ . By our choice of  $C^*$ , the set  $L$  is non-empty.

Let  $H$  be the graph obtained from  $G$  by replacing each edge  $f = vw \in L$  with a loop-edge at  $w$ . By Lemma 3.7,  $H$  is connected. Note that  $H$  is framework for  $M$ . However, this contradicts our choice of  $G$ .  $\square$

We are now ready to prove Theorem 1.4.

**Theorem** (Theorem 1.4 restated). *Let  $M$  be a 3-connected representable matroid. If  $M$  is quasi-graphic, then either  $M$  is a frame matroid or  $M$  is a lifted-graphic matroid.*

*Proof.* Let  $M = M(A)$ , where  $A$  is a matrix over a field  $\mathbb{F}$  with linearly independent rows. We may assume that  $|M| \geq 4$ . Therefore, by Lemma 7.1,  $M$  has a strong framework  $G$ .

**Claim.** *There is a matrix  $B \in \mathbb{F}^{V(G) \times E(G)}$  such that*

- *the row-space of  $B$  is contained in the row-space of  $A$ , and*
- *for each  $v \in V(G)$  and non-loop edge  $e$  of  $G$ , we have  $B[v, e] \neq 0$  if and only if  $v$  is incident with  $e$ .*

*Proof of claim.* Let  $v \in V(G)$  and let  $C^* = E(M) - \text{cl}_M(E(G - v))$ . By the definition of a strong framework,  $C^*$  is a cocircuit of  $M$ . Since  $r(E(M) - C^*) < r(M)$ , by applying row-operations to  $A$  we may assume that there is a row  $w$  of  $A$  whose support is contained in  $C^*$ . Since  $C^*$  is minimally co-dependent, the support of row- $w$  is equal to  $C^*$ . Now we set the row- $v$  of  $B$  equal to the row- $w$  of  $A$ .  $\square$

Note that  $M(B)$  is a frame matroid and  $G$  is a framework for  $M(B)$ . We may assume that  $r(M(A)) > r(M(B))$  since otherwise  $M(A)$  is a frame matroid. Since  $G$  is a connected framework for both  $M(A)$  and  $M(B)$ , it follows that  $r(M(B)) = |V(G)| - 1$  and that  $r(M(A)) = |V(G)|$ . Up to row-operations we may assume that  $A$  is obtained from  $b$  by appending a single row. By Lemma 1.1,  $M(B) = M(G)$ . Hence  $M$  is a lift of  $M(G)$ .  $\square$

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